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Non-singular locally optimal ellipsoidal approximation of the estimate of the states of linear systems☆

A.M. Shmatkov

Moscow, Russia

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ABSTRACT

A problem which arises when estimating the attainability domains of linear dynamical systems by ellipsoids is investigated in a short time interval in the case when the initial position of the system in phase space is known precisely for some at least coordinates. A method is proposed which allows one to avoid problems associated with the degeneracy of the right-hand sides of the differential equations of the locally optimal ellipsoidal approximation. The mathematical meaning of these equations is made more precise in the case of the minimization of the phase volume. An example is given.

1. Introduction

Consider a dynamical system of the form

$$\dot{x} = A(t)x + B(t)\eta(t) + f(t), \quad x(t_0) \in X_0, \quad x \in R^n, \quad \eta(t) \in R^m \tag{1.1}$$

Here, η (t) is uncontrolled disturbance that is limited in magnitude in a known manner. We will assume that all the time functions considered are such that solutions of the differential equations, in which these functions are used, exist and all cases of the imposition of additional constraints will be specified separately. Since nothing more is known about the disturbance $\eta(t)$, it is appropriate to use guaranteed estimation in order to obtain information about the function $x(t)$. The construction of domains of attainability is rather complicated in the multidimensional case, un view of which, we employ the well known method of estimates of these domains using ellipsoids (see Refs. 1–3).

$$E(\chi, D) = \{y \in R^n; (D^{-1}(y-\chi), y-\chi) \le 1\} \tag{1.2}$$

where χ is the centre and D is the matrix of the ellipsoid. An ellipsoid (1.2) with a matrix $Q(t)$ can then be found which contains the entire set of possible values of $x(t)$ and is governed by the equations

$$\dot{Q} = AQ + QA^T + \frac{Q}{q(t)} + q(t)G(t), \quad G(t) = BK(t)B^T, \quad Q(T_0) = Q_0 \tag{1.3}$$

Here $K(t)$ is the matrix of the ellipsoid (1.2) that limits the disturbance and the ellipsoid with the matrix Q_0 limits the set of possible initial values X_0. Without loss of generality, it can be assumed that the centre of the ellipsoid K is located at the origin of the system of coordinates. The motion of the centre of the ellipsoid Q is then described by the equation

$$\dot{a} = Aa + f, \quad a(t_0) = a_0 \tag{1.4}$$

where a_0 is the centre of the ellipsoid Q_0.

The scalar function $q(t)$ is subject to two conditions. Firstly, the differential equation (1.3) must have a solution and, secondly, $q = q(t) > 0$ when $t > t_0$. Existing methods for choosing of the function $q(t)$ can be divided into two groups.[4] In both cases, a functional $L = L(Q)$, which is

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E-mail address: shmatkov@ipmnet.ru.

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smooth and monotonically dependent on Q, is introduced. In other words, it is required that the gradient $\partial L/\partial Q$ should exist and it should be a positive semidefinite. The volume $L(Q)=\mathrm{Vol}E(Q)$ and the trace $L(Q)=\mathrm{Tr}Q$ can serve as examples of functionals. Then,

$$q = (\mathrm{Tr}(PQ)/\mathrm{Tr}(PG))^{1/2} \tag{1.5}$$

where, in the locally optimal case,

$$P(t) = \frac{\partial L}{\partial Q}(t) \tag{1.6}$$

and, in the globally optimal case,

$$\dot{P} = -PA - A^T P, \quad P(T) = \left.\frac{\partial L}{\partial Q}\right|_{Q=Q(T)} \tag{1.7}$$

Version (1.6) means that the rate of growth of the functional is minimized at each instant between t_0 and the instant T when the process is terminated. Version (1.7) ensures a minimum value of the functional at the instant T. Note that the two varerions are identical when $t_0=T$.

In engineering, the linear approximation (1.1) is permissible, as a rule, either in a fairly short time interval or when investigating of the deviation of the actual trajectory from the basic trajectory for comparatively small magnitudes of the vector x. Consequently, the possibility of a rapid solution of the problem of the estimating of the phase state for small value of T and small eigenvalues of the matrix Q_0 is of paramount interest. Version (1.6) is then preferable since, in the general case, the solution of the locally optimal problem is considerably less tedious compared with the solution of the globally optimal problem since the latter is a boundary value problem. Moreover, the solutions of both problems are similar when t_0 and T are close. On the other hand, if the matrix Q_0 is singular, then the calculation of $Q(t)$ is made more difficult in the case of (1.6), since q can vanish at the instant t_0. The position becomes even more complicated if the matrix G is also singular. In particular, the latter case arises naturally in the treatment of mechanical problems in which a force acting on a system serves as an uncertain factor. Then, the components of the matrix G, occurring in the equations which are solved for the derivatives of the coordinates, can be zero, unlike the components occurring in the equations that are solved for the derivatives of the momenta.

2. Formulation of the problem

We will show that the ability to solve the locally optimal problem in the case of a null matrix Q_0 can prove to be useful even in the case when the set of initial conditions is non-degenerate. We take any point belonging to the set X_0 and assume that, at the instant t_0, the system is located precisely at this point. An ellipsoid with the matrix $Q_*(t)$, which is obtained as the solution of Eqs. (1.3), (1.5) and (1.6), with the condition $Q_0=0$, will then be the locally optimal estimate for $x(t)$. The matrix that has been found will be the same for any other initial point. Since the ellipsoid with the matrix Q_0 restricts the set X_0, the totality of the centres of all of the ellipsoids obtained in this a way will itself be an ellipsoid $Q_A(t)$ with a centre which is subject to relation (1.4). The matrix $Q_A(t)$ can be found from the equation

$$\dot{Q}_A = AQ_A + Q_A A^T, \quad Q_A(t_0) = Q_0 \tag{2.1}$$

In other words, the required estimate of the vector $x(t)$ is the union of the ellipsoid the parameters of which are obtained from Eqs. (1.4) and (2.1), with the set of ellipsoids that have the common matrix $Q_*(t)$ and centres at each point of the ellipsoid $Q_A(t)$. According to the definition of the sum of sets, this set is the sum of two ellipsoids: an ellipsoid with a matrix $Q_A(t)$ and centre $a(t)$ and an ellipsoid with a matrix $Q_*(t)$ and centre at the origin of the system of coordinates. This sum can be approximated by a single ellipsoid using well known methods.

Consequently, if it is necessary to solve the problem of the approximating of the phase state of the same dynamical system many times for different initial conditions, it is sufficient to solve a non-linear differential equation of the type (1.3) once for null initial conditions. The linear Eq. (2.1) has to be solved for each version of the other initial conditions and the resulting ellipsoid has to be combined with the solution of Eq. (1.3). Note that the final operation in approximating of the sum may turn out to be optional in an actual application since the set that represents the sum of the two ellipsoids can be used directly.

The proposed method is insensitive to the type of singularity of the initial set, since the right-hand side of Eq. (2.1) does not have singularities in the case of any singularity of the initial matrix. This is important since it has been noted[5] regarding Eq. (1.3), as applied to the locally optimal minimization of the logarithm of the volume, that “a solution of the Cauchy problem does not generally exist for it for the majority of singular initial conditions”

It follows from what has been said above that it is necessary to be able to solve the following Cauchy problem

$$\dot{Q} = A(t)Q + QA^T(t) + \frac{Q}{q(t)} + q(t)G(t), \quad Q(0) = 0 \tag{2.2}$$

for small t, choosing $q(t)>0$, when $t>0$, by the best method in a certain sense.

3. The “Universal” asymptotics relation

We will now construct an approximate solution of Eq. (2.2) in the neighbourhood of the point $t=0$ in the form of series in the small parameter t. Suppose representations

$$A(t) = A_0 + tA_1 + O(t^2), \quad G(t) = G_0 + tG_1 + t^2 G_2 + O(t^3)$$

exist, where A_0, A_1, G_0, G_1, G_2 are known constant matrices.

We shall seek the coefficients of the expansion for $Q(t)$

$$Q(t) = tQ_1 + t^2Q_2 + t^3Q_3 + t^4Q_4 + O(t^5) \tag{3.1}$$

while simultaneously prescribing the coefficients in the expansion of the function

$$q(t) = q_0 + tq_1 + t^2q_2 + O(t^3)$$

We substitute series (3.1) into relation (2.2), transfer all the terms to the left-hand side of the equation, and find the coefficients of the different powers of t. From the condition that the coefficient of t^0 is equal to zero, we obtain $Q_1 = q_0G_0$. Consequently, it is convenient to choose $q_0 = 0$. Equating the coefficient of t^1 to zero, we obtain $Q_2 = q_1/(2 - q_1^{-1})G_0$. We now choose $Q_1 > 0$ such that the coefficient of G_0 has the smallest possible positive magnitude. Then, $q_1 = 1$ and $Q_2 = G_0$. It can be shown that, when $q_0 = 0$ and $q_1 = 1$, the matrix $Q_3 = (A_0G_0 + G_0A_0^T + G_1)/2$. It is independent of terms in the series for the function $q(t)$ of the order of t^2. We obtain

$$Q_4 = \frac{1}{3}\left(A_0Q_3 + A_1Q_2 + Q_3A_0^T + Q_2A_1^T + G_2 + q_2^2G_0 + \frac{q_2}{2}(G_1 - A_0G_0 - G_0A_0^T)\right)$$

Suppose $q_2 = 0$. The relation for $Q(t)$ acquires the form

$$Q(t) = t^2G_0 + t^3Q_3 + \frac{t^4}{3}(A_0Q_3 + Q_3A_0^T + A_1G_0 + G_0A_1^T + G_2) + O(t^5) \tag{3.2}$$

No functional for searching for the optimal form of $q(t)$ was used in deriving of this relation. At the same time, this result possesses a number of useful properties.

1°. The expression

$$Q(t) = t^2G_0 + t^3Q_3 + O(t^4) \tag{3.3}$$

was used[6] for the case of a positive-definite matrix $G(t)$ when searching for the solution that is locally optimal with respect to the logarithm of the volume. It was shown that the same relation under the same conditions holds for an ellipsoid that is guaranteed to be contained in the attainable set. Consequently, in the case of a positive-definite matrix $G(t)$, the attainable set is an ellipsoid with matrix (3.3) apart from terms of the order of t^4.

2°. It follows from known results[4] that, when constructing of globally optimal ellipsoids for any criterion

$$q(t) = \varphi/\dot{\varphi}; \quad \varphi = \sqrt{\mathrm{Tr}(PQ)}, \quad \dot{\varphi} = \sqrt{\mathrm{Tr}(PG)} \tag{3.4}$$

It follows from relations (3.4) that, if the series $\varphi(t) = \varphi(0) + t\dot{\varphi}(+) + O(t^2)$ exists and $\dot{\varphi}(0) \neq 0$, then, when $Q(0) = 0$, we have $\varphi(0) = 0$. Then $q(t) = t + O(r^2)$, and we again obtain expression (3.3).

3°. If we take $q(t) = t$, then the change of variables $Q(t) = tZ(t)$ in problem (2.2) leads to the problem

$$\dot{Z} = A(t)Z + ZA^T(t) + G(t), \quad Z(0) = 0 \tag{3.5}$$

which is linear in Z.

4°. In certain cases involving the combined use of the methods of guaranteed and stochastic estimation, it has been shown[7] that relation (3.2) enables one to improve the estimate of the phase state of a dynamical system.

So, the proposed asymptotic relation can be used in a wide class of engineering problems in which the choice of the specific criterion for the optimality of the estimate is not clear from the physical meaning of the problem or the use of this criterion is not absolutely necessary.

On the other hand, the choice $q(t) = t$ does not, of course, make it possible to reach an extremum in the majority of cases. We will now consider a simple example.

4. The motion of a point mass along a line under the action of an uncertain force

In reduced variables, the corresponding equations for the phase coordinates have the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad |u| \le 1, \quad x_1(0) = x_2(0) = 0 \tag{4.1}$$

Problem (2.2) then takes the form

$$\dot{Q}_{11} = 2Q_{12} + \frac{1}{q(t)}Q_{11}, \quad \dot{Q}_{12} = Q_{22} + \frac{1}{q(t)}Q_{12}, \quad \dot{Q}_{22} = \frac{1}{q(t)}Q_{22} + q(t)$$

$$Q_{11}(0) = Q_{12}(0) = Q_{22}(0) = 0 \tag{4.2}$$

We shall seek the solution which is locally optimal with respect to the area of the ellipse $\mathrm{Vol}\,E(Q) = \pi\sqrt{\det Q}$ in the neighbourhood of the point $t = 0$.

We use the same method as in the preceding section: we put

$$q(t) = q_0 + tq_1 + t^2q_2 + t^3q_3 + O(t^4)$$

If $q_0 \neq 0$, then $\det Q = q_0^2t^4/12 + O(t^5)$. Consequently, it is necessary to choose $q_0 = 0$. We then have

$$Q(t) = t^2Q_2 + t^3Q_3 + t^4Q_4 + O(t^5)$$

$$Q_2 = \mathrm{diag}\left(0, \frac{q_1^2}{2q_1-1}\right), \quad Q_3 = \begin{Vmatrix} Q_3^{11} & Q_3^{12} \\ Q_3^{21} & Q_3^{22} \end{Vmatrix}, \quad Q_4 = \begin{Vmatrix} Q_4^{11} & Q_4^{12} \\ Q_4^{21} & Q_4^{22} \end{Vmatrix}$$

$$Q_3^{11} = 0, \quad Q_3^{12} = Q_3^{21} = \frac{q_1^3}{(3q_1-1)(2q_1-1)}, \quad Q_3^{22} = \frac{2q_2(q_1-1)}{q_1^2}$$

$$Q_4^{11} = 2\frac{q_1^4}{\Delta}, \quad Q_4^{12} = Q_4^{21} = \frac{q_2(11q_1^3 - 22q_1^2 + 12q_1 - 2)}{q_1\Delta}$$

$$Q_4^{22} = \frac{q_2^2(q_1^3 - 4q_1^2 + 6q_1 - 2) + 2q_1^4q_3(q_1-1)}{q_1^3(2q_1-1)(4q_1-1)}$$

$$\det Q = \frac{q_1^6t^6}{(3q_1-1)\Delta} + O(t^7), \ \Delta = (4q_1-1)(3q_1-1)(2q_1-1) \tag{4.3}$$

Since minimization of the rate of growth of the area of the ellipse is equivalent to minimization of the rate of increase of detQ, it suffices to find the minimum with respect to q_1 of the first term of the expansion of detQ, that is,

$$\frac{q_1^6}{(3q_1-1)\Delta} \to \min_{0 < q_1 < +\infty}$$

It can be shown that the required value of q_1 is one of the roots of the equation

$$24q_1^3 - 43q_1^2 + 21q_1 - 3 = 0$$

We obtain

$$q_1 \approx 1.0990, \ Q_{11} \approx 0.312t^4, \ Q_{12} \approx 0.482t^3, \ Q_{22} \approx 1.008t^2 \tag{4.4}$$

Application of the results of the preceding section gives

$$q_1^r = 1, \quad Q_{11}^r = \frac{1}{3}t^4, \quad Q_{12}^r = \frac{1}{2}t^3, \quad Q_{22}^r = t^2 \tag{4.5}$$

The area of the ellipse (4.5) exceeds the area of the ellipse (4.4) by less than 0.8%. Consequently, use of the equality $q(t) = t$ for small t in the problem solved does not lead to significant errors. Note that formulae (4.5) were derived[8] for the case of the local optimization criterion $L(Q) = Q_{22}$.

The results presented do enable one misunderstanding associated with this problem to be cleared up. A solution of problem (4.1), (4.2) was obtained[3] having the form

$$q_1^f = \frac{2}{3}, \quad Q_{11}^f = \frac{32}{45}t^4, \quad Q_{12}^f = \frac{8}{9}t^3, \quad Q_{22}^f = \frac{4}{3}t^2 \tag{4.6}$$

The area of the corresponding ellipse is equal to $S \approx 1.249t^3$ while the area in the case of (4.4) is equal to $S \approx 0.9001t^3$. We will now explain the reason for this discrepancy.

5. Equations of ellipsoids that are locally optimal with respect to the logarithm of the volume

Consider the matrix differential equation

$$\dot{X} = W(t)X, \quad X(t) \in R^{n\times n}, \quad t \in [t_0, T] \tag{5.1}$$

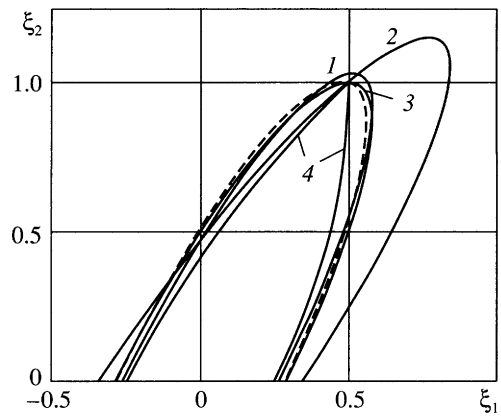

Fig. 1.

It is well known (see Ref. 9, for example) that the Jacobian identity

$$\det X = c\exp\left(\int_{t_0}^{t}\mathrm{Tr}\,W dt\right)$$

where c is a certain constant, holds for system (5.1). Hence we obtain by differentiation,

$$\frac{d}{dt}\ln\det X = \mathrm{Tr}\,W(t) \tag{5.2}$$

In the special case, when $W(t)\equiv X^{-1}(t)\dot{X}(t)$, relation (5.2) becomes[3]

$$\frac{d}{dt}\ln\det X = \mathrm{Tr}(X^{-1}(t)\dot{X}(t)) \tag{5.3}$$

We multiply both sides of the first equation of (1.3) by Q^{-1} and obtain

$$\mathrm{Tr}(Q^{-1}\dot{Q}) = 2\mathrm{Tr}A + \frac{1}{q}\mathrm{Tr}I + q\mathrm{Tr}(Q^{-1}G) \tag{5.4}$$

where I is the unit matrix such that $\mathrm{Tr}I = n$. The property $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$, which is true for any square matrices M and N, has been used in deriving Eq. (5.4) (see Ref. 3). We will now find the minimum of the right-hand side of equality (5.4) for to all $q > 0$. We obtain

$$q = \sqrt{\frac{n}{\mathrm{Tr}(Q^{-1}G)}} \tag{5.5}$$

which is identical to the well-known result[3]. The difference lies in the fact that earlier[3] the discussion concerned ellipsoids that were locally optimal with respect to volume. However, by virtue of the equality (3.3), formula (5.5) ensures the condition $(d/dt)\ln\det Q \to \min$ rather than $(d/dt)\det Q \to \min$.

Thus, the ellipsoids obtained earlier[3] are locally optimal with respect to the logarithm of the volume. Solution (4.6) is actually optimal in this sense as is the other solution that ensures a rate of change of the logarithm of the determinant equal to $6/t$. In particular, solutions (4.4) and (4.5) have precisely this rate.

For completeness, the ellipse obtained earlier,[4] which is globally optimal with respect to its area, should be mentioned. It has the parameters

$$Q^*_{11} = \nu\mu^2 t^4,\; Q^*_{12} = \frac{\nu}{2}t^3,\; Q^*_{22} = \nu t^2,\; \nu = \frac{2\mu^2}{9(\mu^2 - 1/4)} \tag{5.6}$$

where $\mu \approx 0.56215$. Its area $S^* \approx 0.8587t^3$ is less than the area of the locally optimal ellipse (4.4) by approximately 5%. The exact attainable set, obtained earlier,[3] has an area of $(2/3)t^3$.

All the approximations indicated above are shown in the self-similar variables $\xi_1 = x_1/t^2$ and $\xi_2 = x_2/t$ in Fig. 1 (see Ref. 3). By virtue of the symmetry, only the upper half-plane $\xi_2 \geq 0$ is shown. The ellipse (5.6), which is globally optimal with respect to area, is denoted by the number 1, the ellipse (4.6), which is locally optimal with respect to the logarithm of the area, is denoted by 2 and the ellipse (4.5), which corresponds to the asymptotic relation proposed in this paper is denoted by 3. The solution (4.4), which is locally optimal with respect to area, is shown by the dashed curve. Unlike all the remaining ellipses, it is extremal in the above-mentioned sense only for small values of

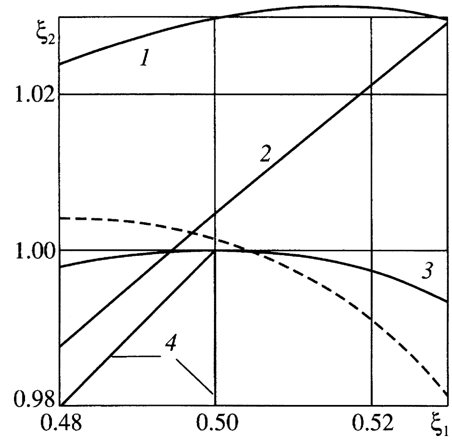

Fig. 2.

t. The approximated attainable set

$$(1+\xi_2)^2/4-1/2 \le \xi_1 \le 1/2-(\xi_2-1)^2/4, \quad |\xi_2| \le 1$$

has a boundary consisting of arcs of two parabolae and is denoted by the number 4. It is interesting that not a single one of the four ellipses completely contains another ellipse but each, as they must, contains the attainable set. Its boundary includes a corner point with the coordinates (0.5, 1), the neighbourhoods of which are shown on a large scale in Fig. 2. It can be seen that only the ellipse (4.5) passes through the above-mentioned point. None of the remaining approximations have common points with the attainable set.

6. Example

We will now consider the motion of two masses with coordinates x_1 and x_2, connected by a spring of stiffness k along a straight line under the action of the limited disturbance F_1 and F_2, where the first of these perturbations acts on the first mass and the second on the second mass respectively. The equations of this mechanical system have the form

$$m_1\ddot{x}_1 = F_1 + k(x_2 - x_1), \quad m_2\ddot{x}_2 = F_2 + k(x_1 - x_2) \tag{6.1}$$

After relations (6.1) have been reduced to normal form, the coordinate and velocity of the first mass will correspond to the first and second variables in phase space, and the coordinate and velocity of the second mass will correspond to the third and fourth variables. In the notation of (1.1), we obtain

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & 0 & k/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/m_2 & 0 & -k/m_2 & 0 \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{Vmatrix}, \quad f = \mathrm{col}(0, 0, 0, 0)$$

Suppose that, at any instant, the magnitudes of the perturbations satisfy the inequality $F_1^2/f_1^2 + F_2^2/f_2^2 \le 1$, where f_1 and f_2 are known constants. Then, in the notation of (1.3), we can write

$$K = \mathrm{diag}(f_1^2, f_2^2), \quad G = \mathrm{diag}(0, f_1^2/m_1^2, 0, f_2^2/m_2^2)$$

We will assume that the following initial conditions are given at the instant $t=0$

$$Q_0 = \mathrm{diag}(0, 0, \alpha^2, \beta^2), \quad a_0 = \mathrm{col}(0, 0, 0, 0)$$

In other words, the coordinate and velocity of the first mass are known absolutely exactly at the initial instant, and the coordinate and velocity of the second mass are known with a certain limited accuracy.

In this example, according to Eq. (1.4), the vector $a(t)$ of the centre of the estimate is identically equal to zero and it is of no interest in the subsequent discussion.

The solution of Eq. (2.1) in the general case has the form (see Ref. 3, for example)

$$Q_A(t) = VQ_0V^T, \quad \dot{V} = A(t)V, \quad V(0) = I \tag{6.2}$$

In the case considered here, A is independent of time. Then, the fundamental matrix $V(t) = \exp(At)$.

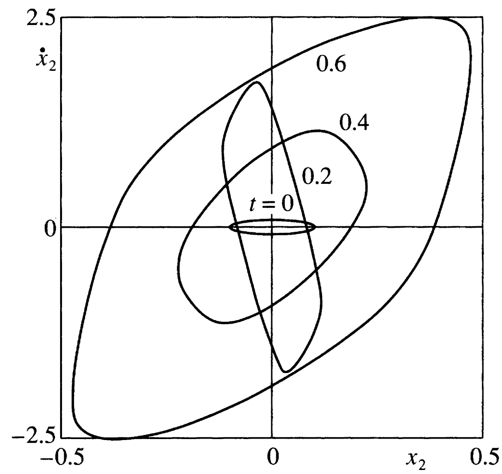

Fig. 3.

The solution of Eq. (2.2) when $q(t) = t$ reduces to the solution of problem (3.5) and, in the general case, it can be written in the form of a quadrature

$$Q(t) = tV(t)\left(\int_0^t V^{-1}(\tau)G(\tau)(V^T(\tau))^{-1}d\tau\right)V^T(t) \tag{6.3}$$

The required approximation is the sum of ellipsoids with matrices $Q_A(t)$ and $Q(t)$, which are found using formulae (6.2) and (6.3) respectively. Both matrices were found in analytical form but cannot be presented in full here because of their length. As an example, we will confine ourselves to the formula

$$Q_{33} = \frac{t}{\omega^7}\left[\frac{1}{2}k_2^2g^2\omega t + \frac{1}{3}(k_1^2g_2^2 + k_2^2g_1^2)\omega^3t^3 - \frac{1}{4}k_2^2g^2\sin(2\omega t) + \right.$$

$$\left. + 2k_2(k_1g_2^2 - k_2g_1^2)(\sin(\omega t) - \omega t\cos(\omega t))\right]$$

$$g^2 = g_1^2 + g_2^2, \quad \omega = \sqrt{k_1 + k_2}, \quad k_j = \frac{k}{m_j}, \quad g_j = \frac{f_j}{m_j}, \quad j = 1, 2$$

From the domain obtained, we separate out that part which bounds the set of possible values of the coordinate and velocity of the second mass at instants when the coordinate and velocity of the first mass are equal to zero. Several sections, labelled with the values of the corresponding instants, are shown in Fig. 3, where all quantities are presented in SI units. Calculations were carried out for the following values of the parameters

$$\alpha = 0.1\text{ m}, \ \beta = 0.1\text{ m s}^{-1}, \ k = 10^3\text{ N m}^{-1}, \ m_1 = 1\text{ kg}, \ m_2 = 2\text{ kg}, \ f_1 = 3\text{ N}, \ f_2 = 5\text{ N}$$

The algorithm used can be developed to obtain approximations in the form of the sum of three or more ellipsoids. We will now consider a certain fixed instant $\tau > 0$. It can be considered as a new origin, and the ellipsoid $Q(t)$, which corresponds to the solution of the problem determined from Eq. (2.2), can be replaced by two ellipsoids. The first will correspond to (2.1) with an initial value equal to $Q(t)$. The second will be the solution of Eq. (2.2), subject to the condition of a time shift by τ. As a result, an estimate will be obtained which represents the sum of these two ellipsoids with the ellipsoid $Q_A(t)$. Similarly, by specifying additional intermediate instants, it is possible to obtain an approximation to any number of ellipsoids specified in advance. It can be used both directly as well as after approximating the unique ellipsoid using well-known formulae.[10]

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